15.2 **Vector Fields**

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$

Sol'n: Gradient field given by $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\ln\left(\sqrt{x^2+y^2+z^2}\right)\right) = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = \frac{x}{x^2+y^2+z^2}$$

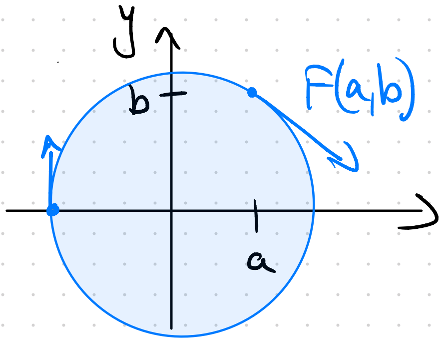
Similarly,

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}, \quad \frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2}.$$

So $$\vec{\nabla} f = \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

15.2 6. Give a formula $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0,0)$ and that at any other point (a,b), $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Sol'n:



$$\vec{F}(x,y) = y\hat{\imath} - x\hat{\jmath}$$

Then at (a,b), $\vec{F}(a,b) = b\hat{\imath} - a\hat{\jmath}$

and $|\vec{F}(a,b)| = \sqrt{b^2 + (-a)^2} = \sqrt{a^2+b^2}$

In polar coordinates, $\vec{F}(r,\theta) = r\sin\theta\,\hat{\imath} - r\cos\theta\,\hat{\jmath}$.

Then we can see that $\vec{F}(r,\theta) \perp r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$ and so is tangent to the circle.

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Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$

7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$

8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$

9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$

10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

Sol'n: $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt.$$

a) $\vec{r}(t) = (t, t, t)$. So $\vec{r}\,'(t) = (1, 1, 1)$.

$$\vec{F}(\vec{r}(t)) = (t\cdot t,\ t\cdot t,\ t\cdot t) = (t^2, t^2, t^2).$$

So $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_0^1 (t^2, t^2, t^2)\cdot(1, 1, 1)\,dt = \int_0^1 3t^2\,dt = t^3\Big|_0^1 = \boxed{1}$$

b) $\vec{r}(t) = (t, t^2, t^4)$. So $\vec{r}'(t) = (1, 2t, 4t^3)$.

$$\vec{F}(\vec{r}(t)) = (t \cdot t^2, t^2 \cdot t^4, t \cdot t^4) = (t^3, t^6, t^5).$$

$$\text{So} \int_C \vec{F} \cdot \hat{T}\, ds = \int_0^1 (t^3, t^6, t^5) \cdot (1, 2t, 4t^3)\, dt$$

$$= \int_0^1 (t^3 + 2t^7 + 4t^8)\, dt = \left(\frac{1}{4}t^4 + \frac{1}{4}t^8 + \frac{4}{9}t^9\right)\Bigg|_0^1$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{4}{9} = \boxed{\frac{17}{18}}$$

c) C_3: $\vec{r}_3(t) = (0,0,0) + t((1,1,0) - (0,0,0)) = (t, t, 0)$, $0 \le t \le 1$

$$\vec{r}_3{}'(t) = (1, 1, 0).$$

C_4: $\vec{r}_4(t) = (1,1,0) + t\big((1,1,1)-(1,1,0)\big) = (1,1,0) + t(0,0,1) = (1,1,t)$;

$0 \le t \le 1$.

$\vec{r}_4{}'(t) = (0,0,1)$.

$\vec{F}(\vec{r}_3(t)) = (t\cdot t, t\cdot 0, t\cdot 0) = (t^2, 0, 0)$.

$\vec{F}(\vec{r}_4(t)) = (1\cdot 1, 1\cdot t, 1\cdot t) = (1, t, t)$.

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_0^1 \vec{F}(\vec{r}_3(t))\cdot\vec{r}_3{}'(t)\,dt + \int_0^1 \vec{F}(\vec{r}_4(t))\cdot\vec{r}_4{}'(t)\,dt$$

$$= \int_0^1 (t^2,0,0)\cdot(1,1,0)\,dt + \int_0^1 (1,t,t)\cdot(0,0,1)\,dt$$

$$= \int_0^1 t^2\,dt + \int_0^1 t\,dt = \frac{1}{3}t^3\Big|_0^1 + \frac{1}{2}t^2\Big|_0^1 = \frac{1}{3}+\frac{1}{2} = \boxed{\frac{5}{6}}$$

15.2 **18.** Along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}$, $0 \le t \le \pi$, evaluate each of the following integrals.

a. $\int_C xz\,dx$ **b.** $\int_C xz\,dy$

c. $\int_C xyz\,dz$

Sol'n:

a) $$\int_C xz\,dz = \int_0^{\pi} \cos t \cdot (-\cos t)\, d(\cos t) = \int_0^{\pi} \cos^2 t \sin t\, dt$$

$$= -\frac{1}{3}\cos^3 t \Big|_0^{\pi} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \boxed{\frac{2}{3}}$$

b) $$\int_C xz\,dy = \int_0^{\pi} \cos t \cdot (-\cos t)\, d(\sin t) = \int_0^{\pi} -\cos^3 t\, dt$$

$$= \frac{1}{12}\left(-9\sin t - \sin(3t)\right)\Big|_0^{\pi} = \boxed{0}$$

c) $$\int_C xyz\,dz = \int_0^{\pi} (\cos t)(\sin t)(-\cos t)\, d(-\cos t)$$

$$= \int_0^{\pi} -\cos^2 t \sin^2 t \, dt = \frac{1}{32}\left(\sin(4t) - 4t\right)\Big|_0^{\pi} = \boxed{-\frac{\pi}{8}}$$

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Work

In Exercises 19–22, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$

Sol'n:

$$W = \int_C \vec{F} \cdot \hat{T}\, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\vec{r}\,'(t) = \left(-\sin t, \cos t, \tfrac{1}{6}\right).$$

$$\vec{F}(\vec{r}(t)) = \left(2\sin t,\ 3\cos t,\ \cos t + \sin t\right).$$

Then

$$W = \int_0^{2\pi} (2\sin t, 3\cos t, \cos t + \sin t) \cdot \left(-\sin t, \cos t, \tfrac{1}{6}\right) dt$$

$$= \int_0^{2\pi} \left(-2\sin^2 t + 3\cos^2 t + \tfrac{1}{6}\cos t + \tfrac{1}{6}\sin t\right) dt$$

$$= \frac{1}{6}\left(3t + \sin t + (15\sin t - 1)\cos t\right)\Big|_0^{2\pi} = \boxed{\pi}$$

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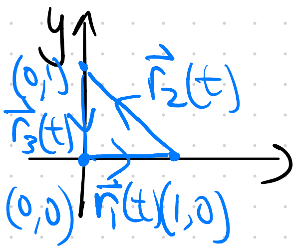
Line Integrals in the Plane

23. Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1,1)$ to $(2,4)$.

24. Evaluate $\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Sol'n:



$C_1: \vec{r}_1(t) = (t,0), \quad 0 \le t \le 1$

$C_2: \vec{r}_2(t) = (1,0) + t((0,1)-(1,0))$

$= (1,0) + t(-1,1) = (1-t,\ t), \quad 0 \le t \le 1$

$C_3: \vec{r}_3(t) = (0,1) + t((0,0)-(0,1)) = (0,1) + t(0,-1) = (0,\ 1-t), \quad 0 \le t \le 1$

$$\int_{C_1} (x-y)\,dx + (x+y)\,dy = \int_0^1 (t-0)\,dt + \cancel{(t+0)\,d(0)}^{\,0} = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

$$\int_{C_2} (x-y)\,dx + (x+y)\,dy = \int_0^1 (1-t-t)\,dt + (1-t+t)\,dt$$

$$= \int_0^1 (1-2t)\,dt + \int_0^1 dt = (t - t^2 + t)\Big|_0^1 = 1$$

$$\int_{C_3} (x-y)\,dx + (x+y)\,dy = \int_0^1 (0-1+t)\,\cancel{d(0)}^{\,0} + (0+1-t)\,d(1-t)$$

$$= \int_0^1 (t-1)\,dt = \left(\frac{1}{2}t^2 - t\right)\Big|_0^1 = -\frac{1}{2}.$$

So $\int_C (x-y)\,dx + (x+y)\,dy = \boxed{1}$

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30. Flux across a circle Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Sol'n: $F_1 = 2x\hat{i} - 3y\hat{j} \Rightarrow M = 2x,\ N = -3y.$

$$\text{Flux of } F_1 = \oint_C M\,dy - N\,dx = \int_0^{2\pi} 2(a\cos t)\,d(a\sin t) + 3(a\sin t)\,d(a\cos t)$$

$$= \int_0^{2\pi} 2a^2\cos^2 t\,dt + \int_0^{2\pi} -3a^2\sin^2 t\,dt$$

$$= 2\pi a^2 - 3\pi a^2 = \boxed{-\pi a^2}$$

$$F_2 = 2x\hat{i} + (x-y)\hat{j} \Rightarrow M = 2x, \quad N = x-y.$$

$$\text{Flux of } F_2 = \oint_C M\,dy - N\,dx = \int_0^{2\pi} 2(a\cos t)\,d(a\sin t) - (a\cos t - a\sin t)\,d(a\cos t)$$

$$= \int_0^{2\pi} 2a^2\cos^2 t\,dt + \int_0^{2\pi} (a^2\cos t\sin t - a^2\sin^2 t)\,dt$$

$$= 2\pi a^2 - \pi a^2 = \boxed{\pi a^2}$$

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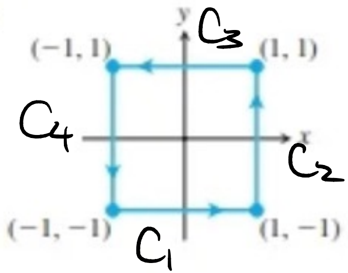
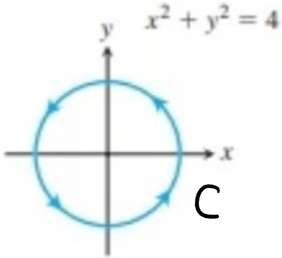
40. Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.

a.



b.



c. Use any closed path different from parts (a) and (b).

a) C_1: $r_1(t) = (-1,-1) + t((1,-1) - (-1,-1)) = (2t-1, -1)$, $0 \le t \le 1$

$r_1'(t) = (2, 0)$.

C_2: $r_2(t) = (1,-1) + t((1,1) - (1,-1)) = (1, 2t-1)$, $0 \le t \le 1$.

$r_2'(t) = (0, 2)$.

$C_3 : r_3(t) = (1,1) + t((-1,1)-(1,1)) = (1-2t, 1)$, $0 \leq t \leq 1$

$r_3'(t) = (-2, 0)$.

$C_4 : r_4(t) = (-1,1) + t((-1,-1)-(-1,1)) = (-1, 1-2t)$, $0 \leq t \leq 1$.

$r_4'(t) = (0, -2)$.

Then circulation $= \int_C \vec{F} \cdot \hat{T}\, ds = \int_{C_1} \vec{F} \cdot \hat{T}\, ds + \int_{C_2} \vec{F} \cdot \hat{T}\, ds + \int_{C_3} \vec{F} \cdot \hat{T}\, ds$

$+ \int_{C_4} \vec{F} \cdot \hat{T}\, ds$.

$$\int_{C_1} \vec{F} \cdot \hat{T}\, ds = \int_0^1 (-1, 2t-1-2) \cdot (2,0)\, dt = \int_0^1 -2\, dt = -2t \Big|_0^1 = -2.$$

$$\int_{C_2} \vec{F}\cdot\hat{T}\,ds = \int_0^1 (2t-1,\ 1+4t-2)\cdot(0,2)\,dt = \int_0^1 (-2+8t)\,dt$$

$$= \left(-2t+4t^2\right)\Big|_0^1 = -2+4 = 2.$$

$$\int_{C_3} \vec{F}\cdot\hat{T}\,ds = \int_0^1 (1,\ 1-2t+2)\cdot(-2,0)\,dt = \int_0^1 -2\,dt = -2.$$

$$\int_{C_4} \vec{F}\cdot\hat{T}\,ds = \int_0^1 (1-2t,\ -1+2-4t)\cdot(0,-2)\,dt = \int_0^1 (-2+8t)\,dt = 2.$$

So $\displaystyle\int_C \vec{F}\cdot\hat{T}\,ds = -2+2-2+2 = 0.$

b) $C:\ r(t) = (2\cos t,\ 2\sin t)$

$r'(t) = (-2\sin t,\ 2\cos t),\qquad 0\leq t\leq 2\pi$

So $\int_C \vec{F}\cdot\hat{T}\,ds = \int_0^{2\pi} (2\sin t,\ 2\cos t + 4\sin t)\cdot(-2\sin t,\ 2\cos t)\,dt$

$= \int_0^{2\pi} (-4\sin^2 t + 4\cos^2 t + 8\cos t\sin t)\,dt$

$= \left(2\sin(2t) - 2\cos(2t)\right)\Big|_0^{2\pi} = 0.$

c) Let C be the circle $x^2+y^2=1$. Then $r(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

$r'(t) = (-\sin t, \cos t)$.

$\int_C \vec{F}\cdot\hat{T}\,ds = \int_0^{2\pi} (\sin t,\ \cos t + 2\sin t)\cdot(-\sin t,\ \cos t)\,dt$

$= \int_0^{2\pi} (-\sin^2 t + \cos^2 t + 2\sin t\cos t)\,dt = \frac{1}{2}\left(\sin(2t) - \cos(2t)\right)\Big|_0^{2\pi} = 0.$

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52. Two "central" fields Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \neq (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is **(a)** the distance from (x, y) to the origin, **(b)** inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

Sol'n: a) At point (x,y), distance to origin is given by $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$.

Take $\vec{F}(x,y) = -x\hat{\imath} - y\hat{\jmath}$. Then $\vec{F}$ points toward the origin with length $|\vec{F}|$ the distance from (x,y) to the origin.

b) Take $\vec{F}(x,y) = \frac{-x}{x^2+y^2}\hat{\imath} - \frac{y}{x^2+y^2}\hat{\jmath}$ where

Then $\vec{F}$ points toward the origin and

$$|\vec{F}(x,y)| = \sqrt{\left(\frac{-x}{(x^2+y^2)}\right)^2 + \left(\frac{-y}{(x^2+y^2)}\right)^2} = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}}$$

as required

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54. Work done by a radial force with constant magnitude A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = k\left[\left(b^2 + (f(b))^2\right)^{1/2} - \left(a^2 + (f(a))^2\right)^{1/2}\right].$$

Sol'n: $r(x) = (x, f(x))$. $a \le x \le b$.

Then $r'(x) = (1, f'(x))$.

F points away from origin, so $\vec{F}(x,y) = cx\hat{i} + cy\hat{j}$ for some constant c

s.t. $|\vec{F}| = \sqrt{c^2x^2 + c^2y^2} = c\sqrt{x^2+y^2} = k$.

$\Rightarrow \vec{F}(x,y) = \dfrac{kx}{\sqrt{x^2+y^2}}\hat{i} + \dfrac{ky}{\sqrt{x^2+y^2}}\hat{j}$.

Then $\displaystyle\int_C \vec{F}\cdot\hat{T}\,ds = \int_a^b \left(\frac{kx}{\sqrt{x^2+(f(x))^2}}, \frac{kf(x)}{\sqrt{x^2+(f(x))^2}}\right)\cdot\left(1, f'(x)\right)dx$

$$= \int_a^b \frac{kx}{\sqrt{x^2+(f(x))^2}} + \frac{kf(x)f'(x)}{\sqrt{x^2+(f(x))^2}}\,dx = k\int_a^b \frac{x+f(x)f'(x)}{\sqrt{x^2+(f(x))^2}}\,dx$$

Note that $\frac{d}{dx}\left(\sqrt{x^2+(f(x))^2}\right) = \frac{1}{2}\left(x^2+(f(x))^2\right)^{-\frac{1}{2}}\left(2x+2f(x)f'(x)\right)$

$$= \frac{x+f(x)f'(x)}{\sqrt{x^2+(f(x))^2}}.$$

So $k\int_a^b \frac{x+f(x)f'(x)}{\sqrt{x^2+(f(x))^2}}\,dx = k\sqrt{x^2+(f(x))^2}\Big|_a^b$

$$= k\left(\left(b^2+(f(b))^2\right)^{\frac{1}{2}} - \left(a^2+(f(a))^2\right)^{\frac{1}{2}}\right)$$ as required